

# Tomographic and Lie algebraic significance of generalized symmetric informationally complete measurements

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Generalized symmetric informationally complete (SIC) measurements are SIC measurements that are not necessarily rank one. They are interesting originally because of their connection with rank-one SICs. Here we reveal several merits of generalized SICs in connection with quantum state tomography and Lie algebra that are interesting in their own right. These properties uniquely characterize generalized SICs among minimal IC measurements although, on the face of it, they bear little resemblance to the original definition. In particular, we show that in quantum state tomography generalized SICs are optimal among minimal IC measurements with given average purity of measurement outcomes. Besides its significance to the current study, this result may help understand tomographic efficiencies of minimal IC measurements under the influence of noise. When minimal IC measurements are taken as bases for the Lie algebra of the unitary group, generalized SICs are uniquely characterized by the antisymmetry of the associated structure constants.

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## I. INTRODUCTION

Quantum state tomography is a primitive of various quantum information processing tasks, such as quantum computation, communications and cryptography. To achieve high tomographic efficiency in practice, it is crucial to choose suitable measurements. Of special interest are the type of measurements that are *informationally complete* (IC) with which every state can be determined completely by the measurement statistics. An IC measurement has at least  $d^2$  outcomes for a  $d$ -level quantum system; those with  $d^2$  outcomes are called *minimal*.

A *symmetric informationally complete* (SIC) measurement [1–8] is composed of  $d^2$  subnormalized projectors onto pure states  $\Pi_j = |\psi_j\rangle\langle\psi_j|/d$  with equal pairwise inner product of  $1/(d+1)$ ,

$$|\langle\psi_j|\psi_k\rangle|^2 = \frac{d\delta_{jk} + 1}{d+1}, \quad j, k = 0, 1, \dots, d^2 - 1. \quad (1)$$

SICs possess many nice properties that make them an ideal choice of fiducial measurements. For example, they are optimal for linear quantum state tomography [9–11] and measurement-based quantum cloning [9]. They play a crucial role in quantum Bayesianism [12–14]. They are also interesting because of their connections with mutually unbiased bases (MUB) [15–19], 2-designs [1, 2, 8, 9], equiangular lines [1, 8], Galois theory [20], Lie algebra [8, 21], adjoint representation of the unitary group [8], and the graph isomorphism problem [6]. Up to now, analytical solutions of SICs and numerical solutions with high precision have been found up to dimension 67 [1–6]. Although SICs are expected to exist for every finite dimension, there is neither a universal construction recipe nor a general existence proof [8].

Generalized SICs are SICs whose outcomes are not necessarily rank one. They were first studied systematically by Appleby [22], and have raised some renewed interest recently [23]. Unlike rank-one SICs, their existence is almost trivial, as the existence of regular tetrahedra, and several explicit construction methods are known [22, 23]. Nevertheless, the study of generalized SICs may promote our understanding about rank-one SICs and provide valuable insight on the SIC existence problem. They are also of interest from a practical point of view since measurements realized in experiments are usually not rank one due to various imperfections, such as noise and dark counts. It is thus highly desirable to determine whether generalized SICs retain some nice properties of rank-one SICs and whether they are optimal for some quantum information processing tasks under such scenarios.

In this paper we reveal several nice properties of generalized SICs in connection with quantum state tomography and Lie algebra. Remarkably, these properties uniquely characterize generalized SICs among minimal IC measurements although they do not bear any resemblance to the original definition. In particular, we show that generalized SICs are optimal in quantum state tomography with minimal IC measurements given the average purity of the measurement outcomes. Besides its significance to the current study, this result is pretty useful in determining the impact of noise on the tomographic efficiency. The outcomes of a minimal IC measurement can also serve as a basis for the Lie algebra of the unitary group. We show that the structure constants associated with this basis are completely antisymmetric if and only if the measurement is a generalized SIC measurement. This observation generalizes the link between SICs and Lie algebra established in Refs. [8, 21]. In the course of our study, we derive several useful results about IC measurements, which may be of independent interest. Our study also leads to an intriguing connection between SICs and MUB popping up in a tomography problem.

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The rest of the paper is organized as follows. In Sec. II we review the basic framework of quantum state tomography. In Sec. III we reexamine tight IC measurements originally introduced by Scott [9]. In Sec. IV we introduce the concept of balanced measurements and propose a conjecture about SICs and MUB. In Sec. V we reveal tomographic significance of generalized SICs as well as their connections with tight IC measurements and balanced measurements. In Sec. VI we present a cute characterization of generalized SICs in terms of Lie algebra. Section VII summarizes this paper.

## II. QUANTUM STATE TOMOGRAPHY

In this section, we review the basic framework of quantum state tomography tailored to the needs of the current work following Refs. [6, 9–11], in preparation for later discussions.

A generalized measurement is composed of a set of outcomes represented mathematically by positive operators  $\Pi_j$  that sum up to the identity 1. Given an unknown state  $\rho$ , the probability of obtaining the outcome  $\Pi_j$  is given by the Born rule:  $p_j = \text{tr}(\Pi_j \rho)$ . Following the convention in Refs. [6, 10, 11], the probability can be expressed as an inner product  $\langle\langle \Pi_j | \rho \rangle\rangle$  between the operator kets  $|\Pi_j\rangle\rangle$  and  $|\rho\rangle\rangle$ , where the double ket notation is used to distinguish them from ordinary kets. A measurement is IC if the outcomes  $\Pi_j$  span the whole operator space.

For an IC measurement  $\{\Pi_j\}$ , there exists a set of reconstruction operators  $\Theta_j$  such that  $\sum_j |\Theta_j\rangle\rangle \langle\langle \Pi_j| = \mathbf{I}$ , where  $\mathbf{I}$  is the identity superoperator. Any state can be recovered from the set of probabilities  $p_j$  using the formula  $\rho = \sum_j p_j \Theta_j$ . In practice, the probabilities  $p_j$  need to be replaced by the frequencies  $f_j$  since the number of measurements is finite. The estimator based on these frequencies  $\hat{\rho} = \sum_j f_j \Theta_j$  is thus different from the true state. Nevertheless, the requirement  $\sum_j |\Theta_j\rangle\rangle \langle\langle \Pi_j| = \mathbf{I}$  on the reconstruction operators guarantees that the estimator is unbiased. The scaled MSE matrix and MSE with respect to the Hilbert-Schmidt (HS) distance of the estimator  $\hat{\rho}$  are given by Refs. [10, 11],

$$\mathcal{C}(\rho) = \sum_j |\Theta_j\rangle\rangle p_j \langle\langle \Theta_j| - |\rho\rangle\rangle \langle\langle \rho|, \quad (2)$$

$$\mathcal{E}(\rho) = \text{Tr}\{\mathcal{C}(\rho)\} = \sum_j p_j \text{tr}(\Theta_j^2) - \text{tr}(\rho^2). \quad (3)$$

Here “Tr” denotes the trace of a superoperator, and “tr” of an ordinary operator.

The set of reconstruction operators is not unique except for a minimal IC measurement. In linear state tomography, usually these operators, once chosen, are independent of the measurement statistics. In that case, the average scaled MSE over unitarily equivalent states

is given by

$$\overline{\mathcal{E}(\rho)} = \frac{1}{d} \sum_j \text{tr}(\Pi_j) \text{tr}(\Theta_j^2) - \text{tr}(\rho^2). \quad (4)$$

*Canonical reconstruction operators*

$$|\Theta_j\rangle\rangle = \frac{d\mathcal{F}^{-1}|\Pi_j\rangle\rangle}{\text{tr}(\Pi_j)} \quad (5)$$

are the best for minimizing the average scaled MSE [6, 9–11], where

$$\mathcal{F} = d \sum_j \frac{|\Pi_j\rangle\rangle \langle\langle \Pi_j|}{\text{tr}(\Pi_j)} \quad (6)$$

is known as the frame superoperator. The minimum reads

$$\overline{\mathcal{E}(\rho)} := \text{Tr}(\mathcal{F}^{-1}) - \text{tr}(\rho^2). \quad (7)$$

If reconstruction operators are allowed to depend on the measurement statistics, the *optimal reconstruction operators* are given by

$$|\Theta_j\rangle\rangle = p_j^{-1} \mathcal{F}(\rho)^{-1} |\Pi_j\rangle\rangle, \quad (8)$$

where

$$\mathcal{F}(\rho) = \sum_j |\Pi_j\rangle\rangle \frac{1}{p_j} \langle\langle \Pi_j| \quad (9)$$

is also called the frame superoperator, which generalizes the definition in Eq. (6) [6, 11]. Unlike usual linear state tomography, the optimal reconstruction operators depend on the unknown state and need to be chosen adaptively in practice. The scaled MSE matrix and MSE turn out to be

$$\mathcal{C}(\rho) = \mathcal{F}(\rho)^{-1} - |\rho\rangle\rangle \langle\langle \rho| = \bar{\mathcal{F}}(\rho)^+, \quad (10)$$

$$\mathcal{E}(\rho) = \text{Tr}\{\mathcal{F}(\rho)^{-1}\} - \text{tr}(\rho^2) = \text{Tr}\{\bar{\mathcal{F}}(\rho)^+\}, \quad (11)$$

where  $\bar{\mathcal{F}}(\rho)$  is the projection of  $\mathcal{F}(\rho)$  onto the space of traceless Hermitian operators and is the superoperator analog of the Fisher information matrix [24];  $\bar{\mathcal{F}}(\rho)^+$  denotes the Moore-Penrose pseudoinverse of  $\bar{\mathcal{F}}(\rho)^+$ , that is, the inverse on its support. Therefore, the above equations actually give the Cramér-Rao bounds [25, 26] for the scaled MSE matrix and MSE [6, 11].

For a minimal IC measurement, the set of reconstruction operators is unique, so Eqs. (10) and (11) reduce to Eqs. (2) and (3) with  $\Theta_j$  canonical reconstruction operators. When  $\rho$  is the completely mixed state, that is  $\rho = 1/d$ , Eqs. (8) and (9) reduce to Eqs. (5) and (6), so the optimal reconstruction is also identical with the canonical reconstruction. The scaled MSE matrix and MSE are respectively given by

$$\mathcal{C}(\rho) = \mathcal{F}^{-1} - \frac{1}{d^2} |1\rangle\rangle \langle\langle 1| = \bar{\mathcal{F}}^+, \quad (12)$$

$$\mathcal{E}(\rho) = \text{Tr}(\mathcal{F}^{-1}) - \frac{1}{d} = \text{Tr}(\bar{\mathcal{F}}^+). \quad (13)$$

### III. TIGHT INFORMATIONALLY COMPLETE MEASUREMENTS

Tight IC measurements were first introduced by Scott [9] as measurements featuring particular simple state reconstruction. Rank-one tight IC measurements are also optimal under linear quantum state tomography and have thus attracted much attention recently [6, 8, 10, 11, 27]. General tight IC measurements are still not well understood, although they are more relevant in real experiments. In this section, we derive several useful results about these measurements, thereby deepening our understanding on this subject. In particular, we determine the minimal average MSE achievable in linear quantum state tomography given the average purity of measurement outcomes and show that the minimum is attained only for tight IC measurements. We also provide an alternative characterization of minimal tight IC measurements, which is quite useful for understanding their structure and their connections with generalized SICs.

A measurement  $\{\Pi_j\}$  is *tight IC* [6, 8–10] if the frame superoperator has the following form

$$\mathcal{F} = d \sum_j \frac{|\Pi_j\rangle\langle\Pi_j|}{\text{tr}(\Pi_j)} = \alpha \mathbf{I} + \beta |1\rangle\langle 1| \quad (14)$$

for some positive constants  $\alpha$  and  $\beta$ . Multiplying the equation by  $|1\rangle\langle 1|$  on the right gives  $\alpha + d\beta = d$ , that is,  $\beta = 1 - \alpha/d$ . Taking trace of the equation yields

$$d^2\alpha + d\beta = d \sum_j \frac{\text{tr}(\Pi_j^2)}{\text{tr}(\Pi_j)} = d^2 \sum_j \frac{\text{tr}(\Pi_j)}{d} \wp_j = d^2 \wp, \quad (15)$$

where  $\wp_j = \text{tr}(\Pi_j^2)/[\text{tr}(\Pi_j)]^2$  is the purity of the outcome  $\Pi_j$  and  $\wp = \sum_j \text{tr}(\Pi_j)\wp_j/d$  can be understood as the average purity of the measurement  $\{\Pi_j\}$ . Both  $\alpha$  and  $\beta$  are functions of the average purity,

$$\alpha = \frac{d^2\wp - d}{d^2 - 1}, \quad \beta = \frac{d^2 - d\wp}{d^2 - 1}. \quad (16)$$

This equation implies that  $\alpha \leq d/(d+1)$  and the inequality is saturated if and only if all  $\Pi_j$  have rank one.

For a tight IC measurement satisfying Eq. (14), the canonical reconstruction operators have a simple form,

$$\Theta_j = \frac{d^2\Pi_j - (d - \alpha)\text{tr}(\Pi_j)}{d\alpha\text{tr}(\Pi_j)}, \quad (17)$$

with

$$\text{tr}(\Theta_j^2) = \frac{d^2\wp_j - d}{\alpha^2} + \frac{1}{d}. \quad (18)$$

According to Eq. (3), the scaled MSE associated with the canonical linear reconstruction is

$$\mathcal{E}(\rho) = \frac{d^2}{\alpha^2} \left[ \wp(\rho) - \frac{1}{d} \right] - \left[ \text{tr}(\rho^2) - \frac{1}{d} \right], \quad (19)$$

where  $\wp(\rho) = \sum_j p_j \wp_j$  is the average purity of the outcomes  $\Pi_j$  weighted by the probabilities  $p_j$ . Note that  $\wp(\rho)$  reduces to the average purity  $\wp$  of  $\{\Pi_j\}$  when  $\rho$  is the completely mixed state; the average of  $\wp(\rho)$  over unitarily equivalent states is also equal to  $\wp$ . Taking average in the above equation yields

$$\overline{\mathcal{E}(\rho)} = \frac{(d^2 - 1)^2}{d^2\wp - d} - \left[ \text{tr}(\rho^2) - \frac{1}{d} \right], \quad (20)$$

where we have applied Eq. (16).

The tomographic significance of tight IC measurements is revealed by the following theorem, which sets the tomographic efficiency limit of linear quantum state tomography in terms of the average purity of measurement outcomes.

*Theorem 1.* In linear quantum state tomography with any IC measurement with average purity  $\wp$ , the average scaled MSE over unitarily equivalent states is lower bounded by Eq. (20). The bound is saturated if and only if the measurement is tight IC.

*Proof.* Let  $\{\Pi_j\}$  be an IC measurement with average purity  $\wp$ . The minimum average scaled MSE under linear tomography is given by  $\text{Tr}(\mathcal{F}^{-1}) - \text{tr}(\rho^2)$  according to Eq. (7). Note that  $\text{Tr}(\mathcal{F}) = d^2\wp$  and that  $|1\rangle\langle 1|$  is an eigenvector of  $\mathcal{F}$  with eigenvalue  $d$ . The minimum of  $\text{Tr}(\mathcal{F}^{-1})$  under these constraints is attained if and only if  $\mathcal{F}$  has the form in Eq. (14), that is, if  $\{\Pi_j\}$  is tight IC. In that case, the lower bound for the average scaled MSE is indeed saturated.  $\square$

The minimum of  $\overline{\mathcal{E}(\rho)}$  is attained when  $\wp = 1$ , that is, when the tight IC measurement is rank one. So rank-one tight IC measurements are optimal in linear quantum state tomography [9–11]. The frame superoperator and the reconstruction operators now simplify to

$$\mathcal{F} = \frac{d}{d+1}(\mathbf{I} + |1\rangle\langle 1|), \quad \Theta_j = (d+1)\frac{\Pi_j}{\text{tr}(\Pi_j)} - 1. \quad (21)$$

The scaled MSE is given by

$$\mathcal{E}(\rho) = d^2 + d - 1 - \text{tr}(\rho^2), \quad (22)$$

which is unitarily invariant. According to Scott [9] (see also Ref. [8]), a rank-one measurement is tight IC if and only if the outcomes form a weighted 2-design. Prominent examples of tight IC measurements are SIC measurements and complete mutually unbiased measurements, that is, measurements composed of complete sets of MUB. In the second example, the scaled MSE can be reduced by the optimal reconstruction [6, 11, 28, 29], with the result

$$\mathcal{E}(\rho) = d^2 + d - (d+1)\text{tr}(\rho^2). \quad (23)$$

It is still unitarily invariant, which is quite rare among informationally overcomplete measurements.

It is not easy to understand the structure of tight IC measurements from the definition. The following lemma

gives an alternative characterization of minimal tight IC measurements, which is useful for understanding their structure and their connections with generalized SICs.

*Lemma 1.* A minimal IC measurement  $\{\Pi_j\}$  is tight IC if and only if it satisfies the equation

$$\text{tr}(\Pi_j \Pi_k) = \tilde{\alpha} \sqrt{\text{tr}(\Pi_j) \text{tr}(\Pi_k)} \delta_{jk} + \tilde{\beta} \text{tr}(\Pi_j) \text{tr}(\Pi_k) \quad (24)$$

for some positive constants  $\tilde{\alpha}$  and  $\tilde{\beta}$ .

*Proof.* Define  $L_j := \Pi_j / \sqrt{\text{tr}(\Pi_j)}$ , then  $\{\Pi_j\}$  satisfies Eq. (14) if and only if  $\{L_j\}$  satisfies

$$d \sum_j |L_j\rangle\langle L_j| = \alpha \mathbf{I} + \beta |1\rangle\langle 1|. \quad (25)$$

According to Theorem 1 in Ref. [8], this equation is equivalent to

$$\text{tr}(L_j L_k) = \tilde{\alpha} \delta_{jk} + \tilde{\beta} \text{tr}(L_j) \text{tr}(L_k), \quad (26)$$

where  $\tilde{\alpha} = \alpha/d$  and  $\tilde{\beta} = \beta/(\alpha + d\beta)$ . Replace  $L_j$  with  $\Pi_j / \sqrt{\text{tr}(\Pi_j)}$  in the equation yields

$$\frac{\text{tr}(\Pi_j \Pi_k)}{\sqrt{\text{tr}(\Pi_j) \text{tr}(\Pi_k)}} = \tilde{\alpha} \delta_{jk} + \tilde{\beta} \sqrt{\text{tr}(\Pi_j) \text{tr}(\Pi_k)}, \quad (27)$$

which is equivalent to Eq. (24). So Eq. (14) is equivalent to Eq. (24) with  $\tilde{\alpha} = \alpha/d$  and  $\tilde{\beta} = \beta/(\alpha + d\beta)$ . Since  $\alpha + d\beta = d$  for a tight IC measurement, it follows that  $\tilde{\beta} = \beta/d$ .  $\square$

#### IV. BALANCED MEASUREMENTS

An IC measurement is *quasi-balanced* if the scaled MSE  $\mathcal{E}(\rho)$  [which is equal to the Cramér-Rao bound; see Eq. (11)] of the optimal reconstruction is unitarily invariant. It is *balanced* if in addition the scaled MSE matrix [see Eq. (12)] at the completely mixed state is invariant under unitary transformations of the measurement outcomes [31]. Intuitively, balanced measurements are those measurements whose tomographic efficiencies are most indifferent to the identity of the true state. This concept also has an intimate connection with operationally invariant information proposed in Ref. [30] (cf. Ref. [28]).

According to Eq. (11), an IC measurement is quasi-balanced if and only if  $\text{Tr}\{\mathcal{F}(\rho)^{-1}\}$  is unitarily invariant. According to Eq. (12), the additional requirement for a balanced measurement is satisfied if and only if the frame superoperator  $\mathcal{F}$  is unitarily invariant. Since the adjoint representation of the unitary group has only two irreducible components, one spanned by the identity operator 1 and the other by traceless Hermitian operators, this requirement is satisfied if and only if  $\mathcal{F}$  has the form as in Eq. (14), that is, if the measurement is tight IC. Therefore, a balanced measurement is one that is both tight IC and quasi-balanced.

According to Sec. III (see also Refs. [6, 11]) and the previous discussion, SIC measurements and complete mutually unbiased measurements are balanced. Actually, they are the only known rank-one balanced measurements with finite number of outcomes (assuming different outcomes of a measurement are not proportional to each other; the covariant measurement is balanced but with infinite number of outcomes). It is plausible that there is a nontrivial connection between SICs and MUB underlying this coincidence.

*Conjecture 1.* SIC measurements and complete mutually unbiased measurements are the only rank-one balanced measurements with finite number of outcomes.

To appreciate the difficulty in constructing balanced measurements, note that combinations of balanced measurements are generally not balanced. For example, the cube measurement in the case of a qubit is not balanced although it is composed of two SIC measurements [11].

In general, it is not easy to characterize all quasi-balanced measurements. For a minimal IC measurement, since the set of reconstruction operators is unique, the minimal scaled MSE  $\mathcal{E}(\rho)$  is determined by Eq. (3), where  $\Theta_j$  are canonical reconstruction operators. It is unitarily invariant if and only if  $\sum_j p_j \text{tr}(\Theta_j^2)$  is unitarily invariant and, consequently, independent of  $\rho$ . This is possible if  $\text{tr}(\Theta_j^2)$  is independent of  $j$  and only then.

*Lemma 2.* A minimal IC measurement is quasi-balanced if and only if all reconstruction operators have the same HS norm.

According to this lemma, any group covariant minimal IC measurement is quasi-balanced since all reconstruction operators have the same spectrum due to symmetry.

For a minimal tight IC measurement, according to Eq. (17) or (18), reconstruction operators have the same HS norm if and only if outcomes have the same purity.

*Lemma 3.* A minimal tight IC measurement is balanced if and only if all outcomes have the same purity.

Alternatively, this lemma follows from Eq. (19).

#### V. TOMOGRAPHIC SIGNIFICANCE OF GENERALIZED SICs

In this section we reveal several tomographic merits of generalized SICs after a short introduction. In particular, we show that among minimal IC measurements, they are identical with balanced measurements and are optimal in quantum state tomography given the average purity of measurement outcomes. Our study generalizes the result of Scott [9] that SICs are optimal minimal IC measurements for linear quantum state tomography.

##### A. Generalized SICs

A measurement  $\{\Pi_j = P_j/d\}$  with  $n$  elements is called a generalized SIC [22, 23] if it is IC and satisfies

$$\text{tr}(P_j P_k) = \alpha \delta_{jk} + \zeta \quad (28)$$

for some real constants  $\alpha$  and  $\zeta$ . The IC requirement implies that the Gram matrix of  $\{P_j\}$  has rank  $d^2$ , so that  $\alpha > 0$  and  $n = d^2$ . Summing over  $k$  in Eq. (28) yields  $\alpha + d^2\zeta = d \operatorname{tr}(P_j) = d$ . So all outcomes  $\Pi_j$  have the same trace of  $1/d$  and the same purity

$$\wp = \frac{\operatorname{tr}(\Pi_j^2)}{[\operatorname{tr}(\Pi_j)]^2} = \operatorname{tr}(P_j^2) = \frac{(d^2 - 1)\alpha + d}{d^2}. \quad (29)$$

Consequently,

$$\alpha = \frac{d^2\wp - d}{d^2 - 1}, \quad \zeta = \frac{d - \wp}{d^2 - 1}. \quad (30)$$

It follows that  $\alpha \leq d/(d+1)$  and the upper bound is saturated if and only if the generalized SIC is rank one. Note that the expression for  $\alpha$  and its range are the same as that for a tight IC measurement; cf. Eq. (16).

According to the above discussion, the outcomes of any generalized SIC can be written as

$$\Pi_j = \frac{1}{d^2}(1 + B_j), \quad (31)$$

where the  $B_j$  form a regular simplex in the space of traceless Hermitian operators. Conversely, any such regular simplex defines a generalized SIC as long as the minimum eigenvalues of  $B_j$  are bounded from below by  $-1$ , as noticed by Appleby [22]. In principle, this observation allows constructing all generalized SICs. For example, any generalized SIC in dimension 2 has the form

$$\Pi_j = x\tilde{\Pi}_j + \frac{1-x}{d^2}, \quad (32)$$

where  $\{\tilde{\Pi}_j\}$  is a rank-one SIC and  $0 < x \leq 1$ . Unfortunately, in general, it is not clear at all whether the set of generalized SICs so constructed contains a rank-one SIC. An alternative construction (with the same limitation) was recently proposed in Ref. [23].

## B. Tomographic significance

According to Lemma 1, any generalized SIC is a tight IC measurement. If the measurement  $\{\Pi_j\}$  satisfies Eq. (28), then it also satisfies Eq. (14) with the same  $\alpha$  and  $\beta = d\zeta$ . Now Lemma 3 implies that it is also a balanced measurement. What is remarkable is that the converse holds for any minimal IC measurement.

*Theorem 2.* A minimal IC measurement is balanced if and only if it is a generalized SIC measurement.

Before proving this theorem, we first point out its main implications. As an immediate consequence, a rank-one minimal IC measurement is balanced if and only if it is SIC. The scaled MSE achievable with a generalized SIC is given by Eq. (20) without the line over  $\mathcal{E}(\rho)$ , that is,

$$\mathcal{E}(\rho) = \frac{(d^2 - 1)^2}{d^2\wp - d} - \left[ \operatorname{tr}(\rho^2) - \frac{1}{d} \right]. \quad (33)$$

Theorems 1 and 2 together imply that

*Theorem 3.* In quantum state tomography with a minimal IC measurement with average purity  $\wp$ , the maximal scaled MSE of any unbiased estimator over unitarily equivalent states is bounded from below by Eq. (33). The bound can be saturated if and only if the measurement is a generalized SIC.

Like Theorem 1, this theorem sets the tomographic efficiency limit in terms of the average purity of measurement outcomes, except that the figure of merit is the maximal scaled MSE instead of the average scaled MSE. Besides, we are not restricted to linear estimators, because the Cramér-Rao bound for the scaled MSE is saturated in canonical linear state tomography with a minimal IC measurement. In addition to providing neat characterizations of tight IC measurements and generalized SIC measurements, the two theorems are quite useful in studying tomographic efficiencies of minimal IC measurements. As an implication of Theorem 3, the maximal scaled MSE with minimal IC measurements is bounded from below by Eq. (22), and the bound can be saturated only for rank-one SIC measurements [9].

According to Lemma 3, to prove Theorem 2, it suffices to show that a minimal tight IC measurement is a generalized SIC measurement if and only if all outcomes have the same purity, which follows from the following lemma.

*Lemma 4.* A minimal tight IC measurement  $\{\Pi_j\}$  is a generalized SIC measurement if and only if any of the following conditions is satisfied:

1.  $\operatorname{tr}(\Pi_j)$  is independent of  $j$ .
2.  $\operatorname{tr}(\Pi_j^2)$  is independent of  $j$ .
3. The purity of  $\Pi_j$  is independent of  $j$ .
4.  $\{\Pi_j\}$  is equiangular.

For a rank-one minimal IC measurement, the purities of all outcomes are automatically identical. Therefore, it is tight IC if and only if it is SIC [8, 9].

*Proof.* Obviously, statements 1, 2, 3, and 4 hold for a generalized SIC. By Lemma 1, the four statements are equivalent for a tight IC measurement, and any of them guarantees that  $\{\Pi_j\}$  is a generalized SIC.  $\square$

## VI. LIE ALGEBRAIC SIGNIFICANCE OF GENERALIZED SICs

The connection between SICs and Lie algebra was first studied by Appleby, Flammia, and Fuchs [21], who showed that the existence of a SIC in dimension  $d$  is equivalent to the existence of a basis for the Lie algebra  $\mathfrak{u}(d)$  such that the structure matrices have a nice  $Q - Q^T$  property. Recently, Appleby, Fuchs, and the author [8] generalized the result by proving that the SIC existence is equivalent to the existence of a basis such that each structure matrix is Hermitian and rank  $2(d-1)$ . The same line of thinking turns out to be useful also for

studying generalized SICs. Here we reveal a cute characterization of generalized SICs in terms of the structure constants of the Lie algebra.

Given a basis  $L = \{L_j\}$  of Hermitian operators for  $\mathfrak{u}(d)$ , the structure constants  $C_{jkl}^L$  and structure matrices  $C_j^L$  for the basis are defined as

$$[L_j, L_k] = \sum_l C_{jkl}^L L_l, \quad (C_j^L)_{kl} = C_{jkl}^L. \quad (34)$$

Note that the structure constants and structure matrices are pure imaginary. The structure matrices are Hermitian if and only if the structure constants are completely antisymmetric [8, 21].

*Theorem 4.* Let  $\{\Pi_j\}$  be a minimal IC measurement serving as a basis for the Lie algebra  $\mathfrak{u}(d)$ . Then the structure constants are completely antisymmetric if and only if  $\{\Pi_j\}$  is a generalized SIC.

*Proof.* If  $\{\Pi_j\}$  is a generalized SIC, then  $\{P_j = d\Pi_j\}$  satisfies the equation

$$\text{tr}(P_j P_k) = \alpha \delta_{jk} + \zeta = \alpha \delta_{jk} + \zeta \text{tr}(P_j) \text{tr}(P_k) \quad (35)$$

for some positive constants  $\alpha$  and  $\zeta$ . Therefore, the structure constants associated with  $\{P_j\}$  are completely antisymmetric according to Lemma 4 in Ref. [8], so are the structure constants associated with  $\{\Pi_j\}$ . If the structure constants are completely antisymmetric, then the same lemma implies that

$$\text{tr}(P_j P_k) = \alpha \delta_{jk} + \zeta \text{tr}(P_j) \text{tr}(P_k) \quad (36)$$

for some positive constants  $\alpha$  and  $\zeta$ . Summing over  $k$  and applying the identity  $\sum_k P_k = d$  yields

$$d \text{tr}(P_j) = \alpha + d^2 \zeta \text{tr}(P_j), \quad (37)$$

which implies that all  $P_j$  have the same trace of 1. So  $\text{tr}(P_j P_k) = \alpha \delta_{jk} + \zeta$  and  $\{\Pi_j\}$  is a generalized SIC.  $\square$

## VII. SUMMARY

We have identified several characteristic merits of generalized SICs in connection with quantum state tomography and Lie algebra. We showed that among minimal IC measurements generalized SICs happen to be balanced measurements, measurements whose tomographic efficiencies are most insensitive to the states under consideration. They are optimal for quantum state tomography given the average purity of measurement outcomes. In addition to establishing the tomographic significance of generalized SICs, our results are expected to play an important role in studying tomographic efficiencies of minimal IC measurements in realistic scenarios. In a different vein, when minimal IC measurements are taken as bases for the Lie algebra of the unitary group, we showed that generalized SICs are uniquely characterized by the antisymmetry of the associated structure constants.

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- [1] G. Zauner, Int. J. Quant. Inf. **9**, 445 (2011).
  - [2] J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, J. Math. Phys. **45**, 2171 (2004), supplementary information including the fiducial kets available at <http://www.cquic.org/papers/reports/>.
  - [3] J. M. Renes, Ph.D. thesis, The University of New Mexico (2004).
  - [4] D. M. Appleby, J. Math. Phys. **46**, 052107 (2005).
  - [5] A. J. Scott and M. Grassl, J. Math. Phys. **51**, 042203 (2010), supplementary information including the fiducial kets available at <http://arxiv.org/abs/0910.5784>.
  - [6] H. Zhu, Ph.D. thesis, National University of Singapore (2012).
  - [7] H. Zhu, J. Phys. A: Math. Theor. **43**, 305305 (2010).
  - [8] D. M. Appleby, C. A. Fuchs, and H. Zhu, *Group theoretic, Lie algebraic and Jordan algebraic formulations of the SIC existence problem* (2014), quantum Inf. Comput. (to be published), arXiv:1312.0555.
  - [9] A. J. Scott, J. Phys. A: Math. Gen. **39**, 13507 (2006).
  - [10] H. Zhu and B.-G. Englert, Phys. Rev. A **84**, 022327 (2011).
  - [11] H. Zhu, Phys. Rev. A **90**, 012115 (2014).
  - [12] C. A. Fuchs and R. Schack, Rev. Mod. Phys. **85**, 1693 (2013).
  - [13] C. A. Fuchs, *QBism, the perimeter of quantum Bayesianism* (2010), available at <http://arxiv.org/abs/1003.5209>.
  - [14] D. M. Appleby, Å. Ericsson, and C. A. Fuchs, Found. Phys. **41**, 564 (2011).
  - [15] I. D. Ivanović, J. Phys. A: Math. Gen. **14**, 3241 (1981).
  - [16] W. K. Wootters and B. D. Fields, Ann. Phys. **191**, 363 (1989).
  - [17] T. Durt, B.-G. Englert, I. Bengtsson, and K. Życzkowski, Int. J. Quant. Inf. **8**, 535 (2010).
  - [18] W. K. Wootters, Found. Phys. **36**, 112 (2006).
  - [19] D. M. Appleby, H. B. Dang, and C. A. Fuchs, *Symmetric informationally-complete quantum states as analogues to orthonormal bases and minimum-uncertainty states* (2007), available at <http://arxiv.org/abs/0707.2071>.
  - [20] D. M. Appleby, H. Yadsan-Appleby, and G. Zauner,

- Quantum Inf. Comput. **13**, 0672 (2013), available at <http://arxiv.org/abs/1209.1813>.
- [21] D. M. Appleby, S. T. Flammia, and C. A. Fuchs, J. Math. Phys. **52**, 022202 (2011).
  - [22] D. M. Appleby, Optics and Spectroscopy **103**, 416 (2007).
  - [23] A. Kalev and G. Gour, *Construction of all general symmetric informationally complete measurements* (2014), URL <http://arxiv.org/abs/1305.6545>.
  - [24] R. A. Fisher, Math. Proc. Cambr. Philos. Soc. **22**, 700 (1925).
  - [25] H. Cramér, *Mathematical Methods of Statistics* (Princeton University Press, Princeton, NJ, 1946).
  - [26] C. R. Rao, Bull. Calcutta Math. Soc. **37**, 81 (1945).
  - [27] A. Roy and A. J. Scott, J. Math. Phys. **48**, 072110 (2007).
  - [28] J. Řeháček and Z. Hradil, Phys. Rev. Lett. **88**, 130401 (2002).
  - [29] F. Embacher and H. Narnhofer, Ann. Phys. **311**, 220 (2004).
  - [30] Č. Brukner and A. Zeilinger, Phys. Rev. Lett. **83**, 3354 (1999).
  - [31] The balanced measurements defined in the author's thesis [6] correspond to quasi-balanced measurements here.